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Rheometrical flow systems
Part 3. Flow between rotating eccentric cylinders

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We give further consideration to flow situations which are steady in the sense that $\partial/\partial t \equiv 0$ but for which individual fluid elements are subjected to a small sinusoidal deformation. The particular situation studied involves the flow between eccentric circular cylinders which rotate about their axes with the same angular velocity Ω. The eccentricity is assumed to be small. It is shown that measurements of the force on the inner cylinder can be used to determine the complex dynamic viscosity of an elastico-viscous liquid.

The theory provides the necessary mathematical background for the operation of a new commercial rheometer. Consideration is given to the possibility of 'on-line' use of such an instrument for control purposes.

1. Introduction

In previous papers under the same title (Walters 1970; Abbott & Walters 1970, hereafter referred to as parts 1 and 2), we considered rheometrical systems in which the flow was steady in the sense that $\partial/\partial t \equiv 0$, while individual fluid elements underwent a small unsteady (sinusoidal) deformation. Particular attention was paid to the Maxwell Orthogonal Rheometer and the Kepes Balance Rheometer. For these instruments, it was shown that certain force or couple measurements could be immediately converted into relevant complex-viscosity data. There is a growing interest in rheometers of this sort for a number of reasons: (i) there is the basic mathematical attraction of novel flow situations; (ii) there is the experimental advantage of requiring a steady flow in the new rheometers instead of the sinusoidal motion in the more conventional instruments; (iii) the new rheometers also have an advantage from the standpoint of interpreting experimental results, since the forces and couples can be immediately converted into relevant dynamic viscosity η' and dynamic rigidity G' data.

In the present paper, we consider a new rheometrical system which makes use of the steady flow between eccentric circular cylinders rotating about their axes with the same angular velocity Ω. We show that the two components of the force on the inner cylinder can be used to determine the dynamic viscosity η' and dynamic rigidity G' of an elastico-viscous liquid. The analysis provides the mathematical background for the measurements made in a new rheometer which is to be manufactured by Sangamo Controls Ltd.

The related problem in viscous-flow theory for which the outer cylinder is stationary has been considered by Wood (1957) and Kamal (1966).

2. Basic theory

All physical quantities will be referred to cylindrical polar co-ordinates (r, θ, z), the z axis being along the axis of the inner cylinder (figure 1). The distance between the axes of the cylinders will be denoted by a. In most of what follows we shall work to first order in a, although second-order terms will be considered in § 4.

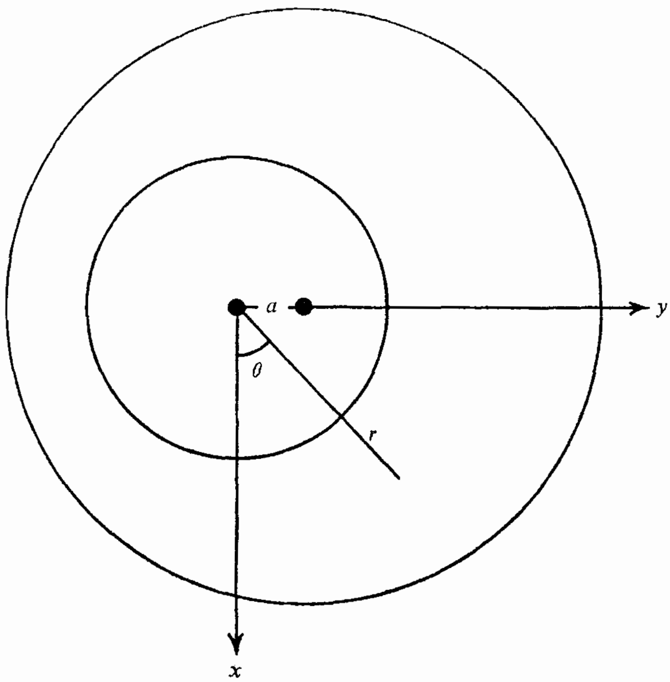

FIGURE 1

If the two cylinders rotate about their axes with the same angular velocity Ω, the relevant boundary conditions to order a are

$$\left.\begin{aligned} &v_{(r)} = 0, \quad v_{(\theta)} = \Omega r_1, \quad v_{(z)} = 0 \quad \text{on} \quad r = r_1, \\ &v_{(r)} = \Omega a\cos\theta, \quad v_{(\theta)} = \Omega[r - a\sin\theta], \quad v_{(z)} = 0 \quad \text{on} \quad r = r_2 + a\sin\theta, \end{aligned}\right\} \tag{1}$$

where r_1 and r_2 are the radii of the inner and the outer cylinders, respectively, and $v_{(r)}$, $v_{(\theta)}$ and $v_{(z)}$ are the physical components of the velocity vector in the r, θ and z directions, respectively.

Incorporating the body forces in the isotropic pressure p, the stress equations of motion for a steady flow with $v_{(z)} \equiv 0$ become†

$$v_{(r)}\frac{\partial v_{(r)}}{\partial r} + \frac{v_{(\theta)}}{r}\frac{\partial v_{(r)}}{\partial \theta} - \frac{v_{(\theta)}^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{\partial p'_{(rr)}}{\partial r} + \frac{\partial}{r\,\partial\theta}p'_{(r\theta)} + \frac{\partial p'_{(rz)}}{\partial z} + \frac{p'_{(rr)} - p'_{(\theta\theta)}}{r}, \tag{2}$$

$$v_{(r)}\frac{\partial v_{(\theta)}}{\partial r} + \frac{v_{(\theta)}}{r}\frac{\partial v_{(\theta)}}{\partial \theta} + \frac{v_{(r)}v_{(\theta)}}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \frac{\partial p'_{(\theta r)}}{\partial r} + \frac{\partial}{r\,\partial\theta}p'_{(\theta\theta)} + \frac{\partial p'_{(\theta z)}}{\partial z} + \frac{2p'_{(\theta r)}}{r}, \tag{3}$$

† Brackets placed round suffices are used to denote the *physical* components of tensors.

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\partial p'_{(zr)}}{\partial r} + \frac{\partial}{r\,\partial \theta} p'_{(z\theta)} + \frac{\partial p'_{(zz)}}{\partial z} + \frac{p'_{(zr)}}{r}, \tag{4}$$

where ρ is the density of the fluid and p_{ik} $(= -pg_{ik} + p'_{ik})$ is the stress tensor, g_{ik} being the metric tensor of the (r, θ, z) co-ordinate system.

The equation of continuity is

$$\frac{\partial}{\partial r}(rv_{(r)}) + \frac{\partial}{\partial \theta}(v_{(\theta)}) = 0. \tag{5}$$

In order to characterize the elastico-viscous liquid by means of suitable equations of state, we note first that when $a = 0$ and the axes of rotation are coincident a rigid body motion exists between the cylinders and the fluid experiences no deformation. The deformation is therefore small provided a is small. We may therefore write the equations of state in the form of integral expansions (cf. Coleman & Noll 1961; Pipkin 1964; parts 1 and 2). The second-order expansion (which is required in §4) is given by†

$$p_{ik} = -pg_{ik} + \int_{-\infty}^{t} M_1(t-t')\,C_{ik}(t')\,dt' + \int_{-\infty}^{t}\int_{-\infty}^{t} M_2(t-t', t-t'')\,C_i^j(t')\,C_{jk}(t'')\,dt'\,dt'', \tag{6}$$

where
$$C_{ik} \equiv \frac{\partial x'^m}{\partial x^i}\frac{\partial x'^s}{\partial x^k} g_{ms}(t') - g_{ik}(t), \tag{7}$$

x'^i being the position at time t' of the element that is instantaneously at the point x^i at time t.

In most of the following analysis, we shall consider only first-order terms in a, so that the relevant expansion for the stress tensor involves just the first integral in (6). Our main concern is the determination of the complex viscosity η^* from the flow situation under consideration, and we note that (cf. parts 1 and 2)

$$\eta^* = \eta' - \frac{iG'}{\Omega} = \frac{i}{\Omega}\int_0^\infty M_1(\xi)[1 - e^{-i\Omega\xi}]\,d\xi. \tag{8}$$

When the axes of rotation are coincident, i.e. when $a = 0$, a solution to the relevant equations exists of the form

$$v_{(r)} = 0, \quad v_{(\theta)} = r\Omega, \quad v_{(z)} = 0. \tag{9}$$

Working to first order in a, the boundary conditions suggest a velocity distribution of the form

$$\left.\begin{aligned} v_{(r)} &= \Omega a F(r) e^{i\theta}, \\ v_{(\theta)} &= \Omega\left[r + ia\frac{d}{dr}\{rF\}e^{i\theta}\right], \\ v_{(z)} &= 0, \end{aligned}\right\} \tag{10}$$

† Covariant suffices are written below, contravariant suffices above, and the usual summation convention for repeated suffices is implied.

where $F(r)$ may be complex and the real part is implied throughout. The velocity distribution (10) satisfies the equation of continuity (5) identically.

To determine the displacement functions x'^i, which we shall write as r', θ', z', we have to solve the equations (Oldroyd 1950)

$$\left.\begin{aligned} \frac{\partial r'}{\partial t} + v_{(r)}\frac{\partial r'}{\partial r} + \frac{v_{(\theta)}}{r}\frac{\partial r'}{\partial \theta} + v_{(z)}\frac{\partial r'}{\partial z} &= 0, \\ \frac{\partial \theta'}{\partial t} + v_{(r)}\frac{\partial \theta'}{\partial r} + \frac{v_{(\theta)}}{r}\frac{\partial \theta'}{\partial \theta} + v_{(z)}\frac{\partial \theta'}{\partial z} &= 0, \\ \frac{\partial z'}{\partial t} + v_{(r)}\frac{\partial z'}{\partial r} + \frac{v_{(\theta)}}{r}\frac{\partial z'}{\partial \theta} + v_{(z)}\frac{\partial z'}{\partial z} &= 0, \end{aligned}\right\} \tag{11}$$

subject to

$$r' = r, \quad \theta' = \theta, \quad z' = z \quad \text{when} \quad t' = t. \tag{12}$$

The solution of (10) and (11) subject to (12) is

$$\left.\begin{aligned} r' &= r + iaF(r)e^{i\theta}[1 - e^{-i\Omega(t-t')}], \\ \theta' &= \theta - \Omega(t-t') - \frac{a}{r}\frac{d}{dr}(rF)\, e^{i\theta}[1 - e^{-i\Omega(t-t')}], \\ z' &= z. \end{aligned}\right\} \tag{13}$$

From (7) and (13), we obtain

$$\left.\begin{aligned} C_{rr} &= 2ia\frac{dF}{dr}e^{i\theta}[1 - e^{-i\Omega(t-t')}], \\ C_{\theta\theta} &= -2iar^2\frac{dF}{dr}e^{i\theta}[1 - e^{-i\Omega(t-t')}], \\ C_{zz} &= 0, \\ C_{r\theta} &= -a\left[r\frac{dF}{dr} + r^2\frac{d^2F}{dr^2}\right]e^{i\theta}[1 - e^{-i\Omega(t-t')}], \\ C_{rz} &= C_{\theta z} = 0. \end{aligned}\right\} \tag{14}$$

We note from the form of the deformation tensor C_{ik} given by (14) that individual fluid elements are subjected to a sinusoidal deformation.

From (6), (8) and (14), we obtain, to first order in a,

$$\left.\begin{aligned} p'_{(rr)} &= 2\eta^*\Omega a\frac{dF}{dr}e^{i\theta}, \\ p'_{(\theta\theta)} &= -2\eta^*\Omega a\frac{dF}{dr}e^{i\theta}, \\ p'_{(zz)} &= 0, \\ p'_{(r\theta)} &= \eta^*\Omega ai\left[r\frac{d^2F}{dr^2} + \frac{dF}{dr}\right]e^{i\theta}, \\ p'_{(rz)} &= p'_{(\theta z)} = 0. \end{aligned}\right\} \tag{15}$$

Substituting (15) into (2) and (3) and writing

$$p = p_0 + \tfrac{1}{2}\rho\Omega^2 r^2 + p_1(r)\eta^*\Omega a\, e^{i\theta},$$

where p_0 is a constant, we obtain

$$\frac{d^2F}{dr^2}+\frac{3}{r}\frac{dF}{dr}=\frac{dp_1}{dr}+\alpha^2\left[2\frac{d}{dr}(rF)-F\right], \tag{16}$$

$$r^2\frac{d^3F}{dr^3}+4r\frac{d^2F}{dr^2}=p_1+\alpha^2\left[2Fr-r\frac{d}{dr}(rF)\right], \tag{17}$$

where (cf. Walters 1968, 1970; Abbott & Walters 1970)

$$\alpha^2=-i\Omega\rho/\eta^*. \tag{18}$$

Eliminating p_1 between (16) and (17), we obtain

$$r^4\frac{d^4F}{dr^4}+6r^3\frac{d^3F}{dr^3}+3r^2\frac{d^2F}{dr^2}-3r\frac{dF}{dr}+\alpha^2\left[r^4\frac{d^2F}{dr^2}+3r^3\frac{dF}{dr}\right]=0, \tag{19}$$

which has to be solved subject to

$$\left.\begin{aligned}F(r_1)=0,\quad F(r_2)=1,\\ F'(r_1)=0,\quad F'(r_2)=0,\end{aligned}\right\} \tag{20}$$

where the dash refers to differentiation with respect to r.

The main applications of the new rheometers under consideration are likely to be in the concentrated polymer solution and polymer melt areas, where inertial effects are negligible. For convenience of presentation, we therefore consider first this important special case and give detailed consideration to inertial effects in the next section.

When inertial effects are ignored, (19) reduces to

$$r^4\frac{d^4F}{dr^4}+6r^3\frac{d^3F}{dr^3}+3r^2\frac{d^2F}{dr^2}-3r\frac{dF}{dr}=0. \tag{21}$$

The solution of (21) subject to (20) is

$$F=Ar^2+B\ln r+\frac{C}{r^2}+D, \tag{22}$$

where

$$\left.\begin{aligned}A&=\frac{2(r_1^2-r_2^2)}{r_1^3r_2^3\Delta},\\ B&=\frac{4(r_2^4-r_1^4)}{r_1^3r_2^3\Delta},\\ C&=\frac{2(r_2^2-r_1^2)}{r_1r_2\Delta},\\ D&=\frac{1}{\Delta}\left[\frac{-2(r_2^2-r_1^2)}{r_1^3r_2^3}+\frac{4(r_1^4-r_2^4)}{r_1^3r_2^3}\ln r_1\right],\end{aligned}\right\} \tag{23}$$

and

$$\Delta=\frac{4}{r_1^3r_2^3}\left[(r_2^4-r_1^4)\ln\frac{r_2}{r_1}-(r_2^2-r_1^2)^2\right].$$

The suggested measurements in the eccentric-cylinder rheometer concern the forces X and Y on the inner cylinder in the x and y directions, respectively (see figure 1). These are given by

$$X=Lr_1\int_0^{2\pi}[p_{(rr)}\cos\theta-p_{(r\theta)}\sin\theta]\,d\theta, \tag{24}$$

$$Y = Lr_1 \int_0^{2\pi} [p_{(rr)} \sin\theta + p_{(r\theta)} \cos\theta]\, d\theta, \tag{25}$$

where L is the length of the column of fluid and the stresses have to be evaluated at $r = r_1$. From (15), (17), (24) and (25), we obtain

$$X - iY = \pi a \Omega \eta^* L \left[-r_1^3 \frac{d^3F}{dr^3} - 3r_1^2 \frac{d^2F}{dr^2} \right], \tag{26}$$

where the derivatives have to be evaluated at $r = r_1$. Substituting (22) and (23) into (26) and separating real and imaginary parts, we obtain

$$X = \frac{4\pi\Omega a \eta' L}{\ln\beta - (\beta^2-1)/(\beta^2+1)}, \tag{27}$$

$$Y = \frac{4\pi a G' L}{\ln\beta - (\beta^2-1)/(\beta^2+1)}, \tag{28}$$

where $\beta = r_2/r_1$. Equations (27) and (28) imply that (X, Ω) and (Y, Ω) measurements can be immediately converted into (η', Ω) and (G', Ω) data. These are likely to be the relevant formulae for most practical applications of the eccentric-cylinder rheometer.

3. Inertial effects

In the case of low viscosity fluids or high rotational speeds, inertial effects may be important. Here, the relevant equation for $F(r)$ is (19), which has to be solved subject to the boundary conditions (20). We first make the substitution

$$H = r\frac{d^2F}{dr^2} + 3\frac{dF}{dr}, \tag{29}$$

which reduces (19) to $$r^2\frac{d^2H}{dr^2} + r\frac{dH}{dr} + (\alpha^2 r^2 - 1)H = 0. \tag{30}$$

The solution of (30) is $$H = A^* J_1(\alpha r) + B^* Y_1(\alpha r), \tag{31}$$

where A^* and B^* are arbitrary constants and J_1 and Y_1 are Bessel functions of the first and second kinds, respectively. Substituting (31) into (29) and solving for F, we obtain

$$F = A^* \int_0^r \frac{1}{\xi_2^3} \int_0^{\xi_2} \xi_1^2 J_1(\alpha\xi_1)\, d\xi_1\, d\xi_2 + B^* \int_0^r \frac{1}{\xi_2^3} \int_0^{\xi_2} \xi_1^2 Y_1(\alpha\xi_1)\, d\xi_1\, d\xi_2 + \frac{\bar{C}}{r^2} + \bar{D}, \tag{32}$$

where $\bar{C}$ and $\bar{D}$ are arbitrary constants. Using results given in Watson (1922, p. 132), (32) reduces to

$$F = \bar{A}\frac{J_1(\alpha r)}{r} + \bar{B}\frac{Y_1(\alpha r)}{r} + \frac{\bar{C}}{r^2} + \bar{D}, \tag{33}$$

where $\bar{A}$ and $\bar{B}$ are arbitrary constants. Applying the boundary conditions (20), we require

$$\bar{A} = \frac{2\alpha}{\bar{\Delta} r_1^3 r_2^3} [r_2^2 Y_2(\alpha r_2) - r_1^2 Y_2(\alpha r_1)], \tag{34}$$

$$\bar{B} = \frac{2\alpha}{\bar{\Delta} r_1^3 r_2^3}[r_1^2 J_2(\alpha r_1) - r_2^2 J_2(\alpha r_2)], \tag{35}$$

$$\bar{C} = \frac{\alpha^2}{\bar{\Delta} r_1 r_2}[J_2(\alpha r_2)Y_2(\alpha r_1) - J_2(\alpha r_1)Y_2(\alpha r_2)], \tag{36}$$

$$\bar{D} = \frac{1}{\bar{\Delta}}\left[\frac{-4}{\pi r_1^3 r_2^3} + \frac{2\alpha}{r_1^4 r_2}[J_2(\alpha r_2)Y_1(\alpha r_1) - J_1(\alpha r_1)Y_2(\alpha r_2)] + \frac{\alpha^2}{r_1^3 r_2}[J_2(\alpha r_1)Y_2(\alpha r_2) - J_2(\alpha r_2)Y_2(\alpha r_1)]\right], \tag{37}$$

where

$$\bar{\Delta} = \frac{-8}{\pi r_1^3 r_2^3} + \frac{\alpha^2}{r_1^3 r_2^3}[r_2^2 - r_1^2][J_2(\alpha r_1)Y_2(\alpha r_2) - J_2(\alpha r_2)Y_2(\alpha r_1)] + \frac{2\alpha}{r_1^4 r_2}[J_2(\alpha r_2)Y_1(\alpha r_1) - J_1(\alpha r_1)Y_2(\alpha r_2)] + \frac{2\alpha}{r_1 r_2^4}[J_2(\alpha r_1)Y_1(\alpha r_2) - J_1(\alpha r_2)Y_2(\alpha r_1)]. \tag{38}$$

Substituting (33) into (26), we obtain

$$X - iY = \pi a \Omega \eta^* L\left[\frac{4\bar{C}}{r_1^2} + \bar{A}\alpha[\alpha^2 r_1^2 J_4(\alpha r_1) - 6\alpha r_1 J_3(\alpha r_1) + 2J_2(\alpha r_1)] + \bar{B}\alpha[\alpha^2 r_1^2 Y_4(\alpha r_1) - 6\alpha r_1 Y_3(\alpha r_1) + 2Y_2(\alpha r_1)]\right], \tag{39}$$

where $\bar{A}$, $\bar{B}$ and $\bar{C}$ are given by (34), (35) and (36).

Equation (39) is likely to be too complicated to be useful to experimentalists who are faced with the problem of interpreting experimental results. However, in most cases where inertial effects are not negligible, they are nevertheless small. It is therefore useful to simplify (39) by considering α to be small. If we use the power series expansions for the Bessel functions in (39) and neglect terms of order α^4, we obtain

$$X - iY = \frac{4\pi\Omega a \eta^* L}{\ln\beta - (\beta^2 - 1)/(\beta^2 + 1)} \times \left[1 + \alpha^2 r_1^2\left\{\frac{12\beta^2(\beta^2+1)\ln\beta - (\beta^2 - 1)(\beta^4 + 10\beta^2 + 1)}{24(\beta^2+1)[(\beta^2+1)\ln\beta - (\beta^2 - 1)]}\right\}\right]. \tag{40}$$

This equation may be compared with the approximate equations which are available in the more conventional methods for measuring η^* (cf. Walters 1968).

If the annular gap is small, it is possible to further simplify (40). In this case, we obtain†

$$X - iY = \frac{12\pi a r_1^3 \Omega \eta^* L}{d^3}\left[1 - \frac{\alpha^2 d^2}{10}\right], \tag{41}$$

where $d = r_2 - r_1$. This formula may be useful to experimentalists in assessing the relative effects of varying the geometrical, flow and material parameters.

† We have simply retained the leading power in d in each relevant term.

4. Non-linear effects

In this section, we show that if terms of order a^2 are included in the analysis, the expression for $X-iY$ is still given by (26), with the implication that non-linear effects will not manifest themselves until terms of order a^3 are considered.

The boundary conditions to order a^2 are

$$\left.\begin{aligned} &v_{(r)} = 0, \quad v_{(\theta)} = \Omega r_1, \quad v_{(z)} = 0 \quad \text{when} \quad r = r_1, \\ &v_{(r)} = \Omega a \cos\theta, \quad v_{(\theta)} = \Omega[\mathrm{r} - a\sin\theta], \quad v_{(z)} = 0, \\ \text{when} \quad & r = r_2 + a\sin\theta - (a^2/2r_2)\cos^2\theta. \end{aligned}\right\} \tag{42}$$

In view of these boundary conditions, we are led to write, to order a^2.

$$v_{(r)} = \Omega[aF(r)e^{i\theta} + a^2M(r)i\,e^{2i\theta}], \tag{43}$$

$$v_{(\theta)} = \Omega\left[r + ai\frac{d}{dr}(rF)\,e^{i\theta} - \tfrac{1}{2}a^2\frac{d}{dr}(rM)\,e^{2i\theta} + a^2N(r)\right], \tag{44}$$

$$v_{(z)} = 0, \tag{45}$$

where M and N are subject to

$$\left.\begin{aligned} M(r) = 0, \quad M'(r) = 0, \quad N(r) = 0 \quad \text{on} \quad r = r_1, \\ M(r) = 0, \quad M'(r) = F'', \quad N(r) = \tfrac{1}{2}rF'' \quad \text{on} \quad r = r_2. \end{aligned}\right\} \tag{46}$$

The velocity distribution given by (43)–(45) automatically satisfies the equation of continuity (5).

For convenience, we shall here neglect inertial effects so that we may assume that F is real (cf. §2). The displacement functions corresponding to (43)–(45) are then given by†

$$\begin{aligned} r' = r &+ iaFe^{i\theta}\{1 - e^{-i\Omega(t-t')}\} \\ &+ a^2\Bigg[\tfrac{1}{2}Me^{2i\theta}\{1 - e^{-2i\Omega(t-t')}\} \\ &+ \frac{F^2e^{2i\theta}}{2r}[-\tfrac{1}{2}\{1 - e^{-2i\Omega(t-t')}\} + \{1 - e^{-i\Omega(t-t')}\}] \\ &+ \left[F\frac{dF}{dr} + \frac{F^2}{2r}\right][-i\Omega(t-t') + \{1 - e^{-i\Omega(t-t')}\}]\Bigg], \end{aligned} \tag{47}$$

$$\begin{aligned} \theta' = \theta &- \Omega(t-t') - \frac{a}{r}\frac{d}{dr}(rF)\,e^{i\theta}\{1 - e^{-i\Omega(t-t')}\} \\ &+ a^2\Bigg[-\frac{i\,e^{2i\theta}}{4r}\frac{d}{dr}(rM)\{1 - e^{-2i\Omega(t-t')}\} - \frac{N(r)}{r}\Omega(t-t') \\ &+ \frac{i}{4}\left[F\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}(rF)\right\} - \frac{1}{r^2}\frac{d}{dr}(rF)\frac{d}{dr}(rF)\right]e^{2i\theta}[\{1 - e^{-2i\Omega(t-t')}\} - 2\{1 - e^{-i\Omega(t-t')}\}] \\ &+ \frac{1}{2}\left[F\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}(rF)\right\} + \frac{1}{r^2}\frac{d}{dr}(rF)\frac{d}{dr}(rF)\right][\Omega(t-t') + i\{1 - e^{-i\Omega(t-t')}\}]\Bigg], \end{aligned} \tag{48}$$

† In this section, we confine attention to elastico-viscous liquids having a fading memory, which implies that the effective range of $(t-t')$ is $(0, \xi)$, where ξ is some multiple of the highest relaxation time. This ensures that the explicit appearance of $(t-t')$ in (47) and (48) does not violate the series expansion in a.

$$z' = z. \tag{49}$$

From (6), (47)–(49), it is not difficult to deduce that the relevant (order a^2) stresses will be of the form

$$\left.\begin{aligned} p_{(rr)} &= a^2[h_1(r)\, e^{2i\theta} + k_1(r)],\\ p_{(\theta\theta)} &= a^2[h_2(r)\, e^{2i\theta} + k_2(r)],\\ p_{(r\theta)} &= a^2[h_3(r)\, e^{2i\theta} + k_3(r)], \end{aligned}\right\} \tag{50}$$

where the h's and k's are in general complex. Substituting (50) into (24) and (25), it can readily be shown that the order a^2 contribution to X and Y will be zero. This means that non-linear effects will not manifest themselves until terms of order a^3 are considered.

The above conclusion, together with a careful study of the experimental results of Maxwell & Chartoff (1965) for the *Orthogonal* Rheometer, leads us to suspect that the working formulae given in §§2 and 3 may be valid over a fairly reasonable range of a, say $0 \leqslant a \leqslant 0{\cdot}3\ (r_2 - r_1)$.

5. On-line possibilities

One of the pressing technological requirements at the present time is for the advent of '*in situ*' rheometers which could provide 'on line' control in industrial processes. Ideally, one would wish to locate such a rheometer in the process without affecting the basic flow. In the case of the eccentric-cylinder rheometer one would therefore envisage a flow along the cylinders (i.e. in the z direction) in addition to the flow created by the rheometer itself. However, this does not appear to be feasible, since the work of Jones (1964) on a related problem would certainly lead one to expect the force in the y direction to be affected by the flow in the z direction, and we shall now show that *both* the forces (i.e. Y *and* X) will in fact be modified by the superimposed flow. To demonstrate this, we still consider the integral equation employed in §§2 and 3 although the deformation is now not necessarily small.†

We consider essentially the same situation as in §2 except that there is now a constant pressure gradient P in the z direction. We therefore consider a velocity distribution of the form

$$\begin{aligned} v_{(r)} &= \Omega a F(r)\, e^{i\theta},\\ v_{(\theta)} &= \Omega\left[r + ia\frac{d}{dr}(rF)\, e^{i\theta}\right],\\ v_{(z)} &= w_0(r) + a w_1(r)\, e^{i\theta}, \end{aligned} \tag{51}$$

where w_0 and w_1 are subject to

$$\left.\begin{aligned} &w_0 = 0 \quad \text{on} \quad r = r_1 \quad \text{and} \quad r = r_2,\\ &w_1 = 0 \quad \text{on} \quad r = r_1,\\ &w_1 = i(dw_0/dr) \quad \text{on} \quad r = r_2. \end{aligned}\right\} \tag{52}$$

† Our concern here is to demonstrate the general effect of the superimposed flow on X and Y and the simple integral equation is sufficient for this purpose.

Solving (11) and (51) for the displacement functions, we obtain

$$r' = r + iaFe^{i\theta}\{1 - e^{-i\Omega(t-t')}\}, \tag{53}$$

$$\theta' = \theta - \Omega(t-t') - \frac{a}{r}\frac{d}{dr}(rF)\, e^{i\theta}\{1 - e^{-i\Omega(t-t')}\}, \tag{54}$$

$$z' = z - w_0(t-t') + \frac{iaw_1}{\Omega} e^{i\theta}\{1 - e^{-i\Omega(t-t')}\} + \frac{aF}{\Omega}\frac{dw_0}{dr} e^{i\theta}[-i\Omega(t-t') + \{1 - e^{-i\Omega(t-t')}\}]. \tag{55}$$

w_0 can be easily obtained from the 'concentric' situation and we shall assume that this is a known function of r in what follows.

From (6) (with $M_2 \equiv 0$), (7), (53)–(55), it can be shown that the stresses to order a are given by

$$p'_{(rr)} = k\left(\frac{dw_0}{dr}\right)^2 + 2\eta^*\Omega a \frac{dF}{dr} e^{i\theta} + 2\mu^* a \frac{dw_0}{dr}\frac{dw_1}{dr} e^{i\theta} + 2ika\frac{d}{dr}\left(F\frac{dw_0}{dr}\right)\frac{dw_0}{dr} e^{i\theta} - 2i\mu^* a \frac{d}{dr}\left(F\frac{dw_0}{dr}\right)\frac{dw_0}{dr} e^{i\theta}, \tag{56}$$

$$p'_{(\theta\theta)} = -2\eta^*\Omega a \frac{dF}{dr} e^{i\theta}, \tag{57}$$

$$p'_{(zz)} = 0, \tag{58}$$

$$p'_{(r\theta)} = \eta^* ia\Omega\left(r\frac{d^2F}{dr^2} + \frac{dF}{dr}\right) e^{i\theta} + \mu^* ia \frac{w_1}{r}\frac{dw_0}{dr} e^{i\theta} - ka\frac{F}{r}\left(\frac{dw_0}{dr}\right)^2 e^{i\theta} + \mu^* a \frac{F}{r}\left(\frac{dw_0}{dr}\right)^2 e^{i\theta}, \tag{59}$$

$$p'_{(rz)} = \eta_0\frac{dw_0}{dr} + \eta^* a \frac{dw_1}{dr} e^{i\theta} + i\eta_0 a \frac{d}{dr}\left(F\frac{dw_0}{dr}\right) e^{i\theta} - i\eta^* a \frac{d}{dr}\left(F\frac{dw_0}{dr}\right) e^{i\theta}, \tag{60}$$

$$p'_{(\theta z)} = i\eta^* a \frac{w_1}{r} e^{i\theta} - \eta_0 a \frac{F}{r}\frac{dw_0}{dr} e^{i\theta} + \eta^* a \frac{F}{r}\frac{dw_0}{dr} e^{i\theta}, \tag{61}$$

where

$$\left.\begin{aligned} \eta_0 &= -\int_0^\infty M_1(\xi)\xi\, d\xi, \\ k &= \int_0^\infty M_1(\xi)\xi^2\, d\xi, \\ \mu^* &= -\frac{i}{\Omega}\int_0^\infty M_1(\xi)\{1 - e^{-i\Omega\xi}\}\xi\, d\xi. \end{aligned}\right\} \tag{62}$$

We note that when $\Omega = 0$, $\mu^* = k$, $\eta^* = \eta_0$, η_0 and k being real constants.

From the equations of motion, the 'order a' contribution to the pressure is given by

$$p = \Omega\eta^* a\, e^{i\theta}\left(r^2\frac{d^3F}{dr^3} + 4r\frac{d^2F}{dr^2}\right)$$
$$+\mu^* a\, e^{i\theta}\left[r\frac{d}{dr}\left(\frac{w_1}{r}\frac{dw_0}{dr}\right) + \frac{2w_1}{r}\frac{dw_0}{dr} - ir\frac{d}{dr}\left\{\frac{F}{r}\left(\frac{dw_0}{dr}\right)^2\right\} - 2i\frac{F}{r}\left(\frac{dw_0}{dr}\right)^2\right]$$
$$+ka\, e^{i\theta}\left[ir\frac{d}{dr}\left\{\frac{F}{r}\left(\frac{dw_0}{dr}\right)^2\right\} + 2i\frac{F}{r}\left(\frac{dw_0}{dr}\right)^2\right]. \qquad (63)$$

Substituting (56), (59) and (63) into (24) and (25), we obtain

$$X - iY = \pi a\Omega\eta^* L\left[-r_1^3\frac{d^3F}{dr^3} - 3r_1^2\frac{d^2F}{dr^2}\right] + \pi a\mu^* L r_1\frac{dw_1}{dr}\frac{dw_0}{dr}. \qquad (64)$$

In principle, F and w_1 can be determined from two coupled differential equations obtained from the equations of motion. These are too complicated to be given here.

From (64) it is not difficult to deduce that when $\Omega = 0$, $X = 0$, as one would expect from symmetry considerations. However, when $\Omega \neq 0$, the fact that $F(r)$ depends on w_0 and w_1 (and hence on the pressure gradient P) and the fact that μ^* is in general complex implies that both X and Y are affected by the superimposed flow, the one exception being when this flow is very slow.

We see, therefore, that for the rheometer to be useful in on-line control it would have to be located in a 'branch' line where the general flow could be stopped during the time of testing. In this connexion, the eccentric-cylinder rheometer would appear to have significant advantages over the orthogonal and balance rheometers considered in parts 1 and 2.

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